The Strong Completeness Theorem for n-Valued Sentential Logic

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1 Introduction

Classical logic has just two truth values, namely *true* and *false*. Any sentence symbol or well-formed formula will take one of these two truth values in any interpretation. Although we are familiar with 2-valued sentential logic, there is no inherent restriction on the number of truth values that can be assigned to a statement. There are other logics with intermediate truth values. In other words, they have a truth value for *true*, one for *false*, and also other values in between.

Let us take the example of the statement "Kurt is tall." The truth or falseness of this statement is dependent on frame of reference and may be more true or more false depending on how tall Kurt is. If Kurt is 5'0, then it might be agreed that the statement "Kurt is tall" is false. If Kurt is 7'0, it might be agreed that the statement is true. However, if Kurt is 6'0, there is more ambiguity.

In a more formal construction, let us say that there are 3 values for the trueness of "tall," *true*, *false*, and *somewhat true*. Then we could represent these as 1, meaning $true$, 0, meaning $false$, and $\frac{1}{2}$, meaning neither tall nor not tall. We could then define an algorithm to assign one of these truth values depending on Kurt's acutal measurement.

Logics with multiple truth values (i.e., more than 2) were studied in the 1920s by several famous Logicians. The earliest references begin in 1920 with a paper by Jan Łukasiewicz where the Polish Logician introduced a logic with 3 truth values, like our example of "Kurt is tall" above. Shortly after, Łukasiewicz, along with the famous Alfred Tarski, extended their work to m truth values. Others have built on the foundational work of these two, including a formalization of 3-valued sentential logic in 1931 by Mordchaj Wajsberg. [?]

Later, beginning in the mid-1940s and extending through the 1950s, J. Rosser and B. Turquette provided extensive work in the field of n-valued sentential logic. It is from Rosser and Turquette's paper, "Axiom Schemes for M-Valued Propositional Calculi" that a cornerstone of the argument we will explore comes. In 1974, H. Goldberg, H. Leblanc, and G Weaver published a paper, “A Strong Completeness Theorem for 3-Valued Logic.” This will provide the structure for the argument here, which blends the process of the latter article with some generalizations of the former. [?]

1.1 Definitions and Terminology

We first need to define a baseline of terminology from which we will work in this paper.

We will use a basic set of connectives, namely $\neg$, $\rightarrow$, and parentheses. We employ a countably infinite set of sentence letters, denoted by S_0, S_1, $\ldots$, and we will use A, B, C, etc. to denote arbitrary sentence letters. As we are used to, the length of our sentence letters will be 1, and we denote this with $lh(S_i) = 1$.

From the sentence letters, we will construct well-formed formulas, which are simply built up from our sentence letters and connectives. We will denote the well-formed formulas with greek letters, ϕ, ψ, etc., and we note that all of our well-formed formulas are either sentence letters themselves, or of the form $\neg\phi$ or $(\phi\rightarrow\psi)$. The length of a well-formed formula in the case of $\neg\phi$ will be $1 + lh(\phi)$ and in the case of $(\phi\rightarrow\psi)$, it will be $3 + lh(\phi) + lh(\psi)$. Sets of well-formed formulas will be noted with capital Greek letters, such as Σ or Γ.

Definition 1. *The set of truth values in n-valued sentential logic is* $\boldsymbol{T}_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$.

Note that in this discussion, there is exactly one value of true, and one value of false, namely 0=*False* and 1=*True*. [1]

The sentence letters, well-formed formulas, and connectives we will use are independent of our n. The set, $\mathbf{T}_n$, of possible truth values in our n-valued sentential logic, is generalized as defined above.

We must note, here, that $\mathbf{T}_n \subseteq \mathbf{T}_m$ if and only if (n-1) divides (m-1). This is an important fact that tells us that if we have two logics, n-valued and m-valued, the set of truth values of n-valued are contained in the set of truth values for m-valued only when n-1 divides m-1. For example, this tells us that in 3-valued logic, we have 3 truth values that are wholly contained within the set of 5 truth values for 5-valued logic, since 3-1=2 divides 5-1=4. We can see this clearly, when we list the set $\mathbf{T}_3=\{0, \frac{1}{2}, 1\}$, and the set $\mathbf{T}_5=\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

Definition 2. *An n-valued truth assignment is a function* $h:\{S_i : i \in \omega\}\rightarrow\boldsymbol{T}_n$. *We extend h to be defined on all well-formed formulas by:*
a) $h(\neg\phi)=1-h(\phi)$
b) $h(\phi\rightarrow\psi)=\min\{1, 1-h(\phi)+h(\psi)\}$

We note that this definition agrees with our standard truth assignents with n=2.

[1] There are other logics that consider multiple values representing true, for example, those where everything over $\frac{1}{2}$ is true. However, the scope of this paper will be limited to one designated truth value.

Definition 3. *ϕ is an n-valued tautology if and only if h(ϕ)=1 for all n-valued truth assignments h. We use the notation $\models^n \phi$ to indicate ϕ is a tautology.*

We note, here, that there are some well-formed formulas that will be tautologies for some values of n, but not others. This is critical, since we need to ensure that we work with validities that are uniform in n.

We next need to define satisfiable and logical consequence.

Definition 4. *A set Σ of well-formed formulas is satisfiable in n-valued sentential logic if there is at least one n-valued truth assignment h such that h(σ)=1 for all $\sigma \in \Sigma$.*

Then, further, we can define a logical consequence in this way:

Definition 5. *ϕ is an n-valued logical consequence of Σ if for any n-valued truth assignment h that satisfies Σ, then h(ϕ)=1. We write $\Sigma \models^n \phi$.*

We can also note that if Σ is n-valued satisfiable, and (n-1) divides (m-1), then we can conclude that Σ is also m-valued satisfiable. The set of truth values for n-valued sentential logic are a subset of the truth values for m-valued sentential logic, since (n-1) divides (m-1), and so a truth assignment h which satisfies Σ in n-vlaued will also satisfy Σ in m-vlaued, since it can assign the same truth values.

Definition 6. *A deductive system is given by a set of logical axioms and rules for deriving more well-formed formulas from those axioms.*

We will use modus ponens as our only rule, which states that from ϕ and $(\phi \rightarrow \psi)$ we can conclude ψ. Note that application of this rule preserves validity due to the following lemma, and, more generally, logical consequence.

Lemma 1. *If $\Sigma \models^n \phi$ and $\Sigma \models^n (\phi \rightarrow \psi)$ then $\Sigma \models^n \psi$.*

We must also define a set of logical axioms Λ_n such that $\models^n \phi$ for all $\phi \in \Lambda_n$.

Definition 7. *A deduction in n-valued logic from a set Σ is a finite sequence $\phi_0, \phi_1, \ldots, \phi_m$ of well-formed formulas such that for each $i \leq m$ one of the following holds:*
(i) $\phi_i \in \Lambda_m \cup \Sigma$
(ii) there are j, $k<i$ such that $\phi_k = (\phi_j \rightarrow \phi_i)$, in other words, ϕ_i follows from ϕ_j, ϕ_k by modus ponens.
We say ϕ is deducible from Σ, written $\Sigma \vdash^n \phi$, if an only if there is a deduction in n-valued logic $\phi_0, \phi_1, \ldots, \phi_m$ from Σ such that $\phi = \phi_m$.

We also use the idea of an inconsistent set.

Definition 8. *A set Σ is said to be* <u>*n-valued inconsistent*</u> *if there exists a well-formed formula ϕ such that $\Sigma \vdash^n \phi$ and $\Sigma \vdash^n \neg\phi$. We say a set is* <u>*consistent*</u> *if the set is not inconsistent.*

Finally, we need to introduce the Soundness Theorem at this point.

Theorem 1. *(Soundness): Let Σ be a set of well-formed formulas and ϕ be a well-formed formula. If $\Sigma \vdash^n \phi$ then $\Sigma \models^n \phi$.*

Proof: Let ϕ_0, ϕ_1, ..., ϕ_m be a deduction of ϕ from our set Σ such that ϕ_n=ϕ.

We want to show that $\Sigma \models^n \phi_i$ for all i$\leq$m. We will use induction on i.

Our base case is ϕ_0, where i=0. We know that $\phi_0 \in \Sigma$ or $\phi_0 \in \Lambda_n$ where Λ_n is our set of logical axioms. If $\phi_0 \in \Sigma$, then $\Sigma \models^n \phi_0$. If $\phi_0 \in \Lambda_n$, then $\models^n \phi_0$, which meams that $\Sigma \models^n \phi_0$.

We will assume our inductive hypothesis, that for all i<m, $\Sigma \models^n \phi_i$. Then, we wish to show that $\Sigma \models^n \phi_m$.

We know that there are three possibilities for ϕ_m; it can be in Σ, a logical axiom, or the conclusion of an implication.

Case 1: If $\phi_m \in \Sigma$ or $\phi_m \in \Lambda_n$, then, as above, $\Sigma \models^n \phi_m$.

Case 2: ϕ_m is be the conclusion of an implication. In other words, ϕ_k= ($\phi_i \rightarrow \phi_m$) and k,i<m. By our inductive hypothesis, then, $\Sigma \models^n \phi_i$ and $\Sigma \models^n \phi_k$=($\phi_i \rightarrow \phi_m$), since k,i<m. Then, by Lemma 1 we have that $\Sigma \models^n \phi_m$. Therefore, if $\Sigma \vdash^n \phi$, then $\Sigma \models^n \phi$. $\square$

1.2 Goal

Our goal is to investigate proof systems for n-valued sentential logic satisfying the Strong Completeness Theorem: $\Sigma \vdash^n \phi$ if and only if $\Sigma \models^n \phi$. We provide here a set of 12 axiom schemas as our Λ_n such that if they are all $\vdash^n$-provable, then strong completeness holds. We do not claim that this Λ_n is minimal, but rather assert that if a minimal set Λ_{min} is obtained, that each of the axioms schemas within Λ_n will be either an axiom within Λ_{min} or provable from the axioms within Λ_{min}. In this way, the proof of Strong Completeness provided here will stand, and further, the proof will be automatic without further work if these axioms are deducible.

2 Definition of $\mathbf{J}_q^n(\phi)$

One of the obstacles in working with n-valued sentential logic is that $\mathbf{T}_n$ is dependent on n, and it is difficult to make a general argument that applies to all n. Rosser and Turquette introduce a concept that we will need as we move forward with our proof of the Completeness Theorem.

Rosser and Turquette claim and prove that for any well-formed formula ϕ in n-valued sentential logic, there is another well-formed formula that we can say is strictly true, taking the truth value of 1, or strictly false, taking the truth value of 0, depending on the truth value of ϕ.

First, the key characteristic of this new formula can be expressed in the following theorem. Note that we express the truth value assigned to the formula using our truth assignment h, as we have already defined it.

Theorem 2. *For any n and any $q \in \boldsymbol{T}_n$ and any well-formed formula ϕ, there is a well-formed formula $J^n_q(\phi)$ such that,*

$$h(J^n_q(\phi)) = \begin{cases} 1 & \textit{iff } h(\phi) = q; \\ 0 & \textit{otherwise.} \end{cases}$$

We think of $J^n_q(\phi)$ as saying "ϕ has truth value q."

We do not provide the proof of this theorem here, since it is rather long, but we do explicitly define $\mathrm{J}^n_1(\phi)$ for all n and $\mathrm{J}^3_q(\phi)$ for each $q \in \mathbf{T}_3$

It is necessary to discuss $\mathrm{J}^n_q(\phi)$ in further detail, and we continue to follow the arguments provided by Rosser and Turquette.

We need to define a notation γ recursively to use in our explanation of $\mathrm{J}^n_q(\phi)$. For any well-formed formula ϕ in our language, let $\gamma_1(\phi)$=$\neg\phi$, and let $\gamma_{i+1}(\phi)$=$\phi\rightarrow\gamma_i(\phi)$.

Let us use our definition of our truth assignment h to explore values of γ.

Claim: h($\gamma_i(\phi)$)=min{1, (1-h(ϕ)·i}.

Proof by induction on i:

If i=1, then h($\gamma_i(\phi)$)=h($\neg\phi$)=1-h(ϕ). Since i=1, this is also equal to (1-h(ϕ))·i. Since 0$\leq$h(ϕ)$\leq$1, then 0$\leq$1-h(ϕ)$\leq$1, and therefore, h($\gamma_i(\phi)$)=min{1, (1-h(ϕ))·i}

Assume that h($\gamma_i(\phi)$)=min{1, (1-h(ϕ))·i} for i$\leq$m. Prove true for i=m+1. $\gamma_{m+1}(\phi)$=$\phi\rightarrow\gamma_m(\phi)$, by the definition of γ. h($\gamma_{m+1}(\phi)$)=min{1, 1-h(ϕ)+h($\gamma_m(\phi)$)} by the definition of our truth assignment h. This equals min{1, 1-h(ϕ)+min{1, (1-h(ϕ))·m}} since h($\gamma_m(\phi)$)=min{1, (1-h(ϕ))·m} by our inductive hypothesis. Then, min{1, 1-h(ϕ)+min{1, (1-h(ϕ))·m}}=min{1, 1-h(ϕ)+(1-h(ϕ))·m} since if min{1, (1-h(ϕ))·m}=1, then min{1, 1-h(ϕ)+min{1, (1-h(ϕ))·m}}=1. This, then, yields our desired result: h($\gamma_{m+1}(\phi)$=min{1, (1-h(ϕ))(m+1)} □

Let us now define $\mathrm{J}^n_1(\phi)$=$\neg\gamma_{n-1}(\phi)$ in n-valued logic. Note that this defines $\mathrm{J}^n_1(\phi)$ for all n.

We claim that if h(ϕ)=1, then h($\mathrm{J}^n_1(\phi)$)=1 and if h(ϕ)$\neq$1, then h($\mathrm{J}^n_1(\phi)$)=0.

We see that by our previous claim, h($\gamma_{n-1}(\phi)$)=min{1, (1-h(ϕ))(n-1)}. Thus, when h(ϕ)=1, h($\gamma_{n-1}(\phi)$)=min{1, (1-1)(n-1)}=0. Since $\mathrm{J}^n_1(\phi)$=$\neg\gamma_{n-1}(\phi)$, we have $\mathrm{J}^n_1(\phi)$=1 and when h(ϕ)$\neq$1, $\mathrm{J}^n_1(\phi)$=0. □

The theorem above states that $J^n_q(\phi)$ is a well-formed formula, built up using sentence letters and $\neg$ and $\rightarrow$.

By way of example, we consider 3-valued sentential logic. Our set of truth values $\mathbf{T}_3=\{0, \frac{1}{2}, 1\}$, so therefore, for a given well-formed formula ϕ, we have 3 explicit values of q in $J^3_q(\phi)$, $J^3_0(\phi)$, $J^3_{\frac{1}{2}}(\phi)$, and $J^3_1(\phi)$.

According to our definition, $J^3_1(\phi)=\neg\gamma_{n-1}(\phi)=\neg\gamma_2(\phi)$. By our recursive definition of $\gamma_i(\phi)$, $\gamma_2(\phi)=\phi\rightarrow\neg\phi$. So, $\neg\gamma_2(\phi)=\neg(\phi\rightarrow\neg\phi)$. As we know, $h(\gamma_i(\phi))=\min\{1, (1-h(\phi))\cdot i\}$. So, $h(J^3_1(\phi))= h(\neg\gamma_2(\phi))=\min\{1, (1-h(\phi))\cdot 2\}$. Since the possible values of $h(\phi)$ are 0, $\frac{1}{2}$, and 1,

$$h(J^3_1(\phi)) = \begin{cases} 1 & \text{if } h(\phi) = 1; \\ 0 & \text{if } h(\phi) = 0 \text{ or } h(\phi) = \frac{1}{2}. \end{cases}$$

The explicit statement of $J^3_0(\phi)=\neg(\neg\phi\rightarrow\phi)$. We can also say from our earlier work, and it is easily verifiable, that

$$h(J^3_0(\phi)) = \begin{cases} 1 & \text{if } h(\phi) = 0; \\ 0 & \text{if } h(\phi) = 1 \text{ or } h(\phi) = \frac{1}{2}. \end{cases}$$

Rosser and Turquette provide a procedure for finding ϕ for each $J^n_q(\phi)$ in n-valued logic. [**?**][*Page*64] Utilizing this procedure for $J^3_{\frac{1}{2}}(\phi)$ yields $J^3_{\frac{1}{2}}(\phi)=J^3_0([(\neg\phi\rightarrow\phi)\rightarrow\phi]\rightarrow\neg[(\phi\rightarrow\neg\phi)\rightarrow\neg\phi])$. Since we said that $J^3_0(\phi)=J^3_1(\neg\phi)$, and $J^3_1(\phi)=\phi\rightarrow\neg\phi$, we can conclude that the explicit statement of $J^3_{\frac{1}{2}}(\phi)=$ $\neg[([(\neg\phi\rightarrow\phi)\rightarrow\phi]\rightarrow\neg [(\phi\rightarrow\neg\phi)\rightarrow\neg\phi])\rightarrow\neg ([(\neg\phi\rightarrow\phi)\rightarrow\phi]\rightarrow\neg[(\phi \rightarrow\neg\phi)\rightarrow\neg\phi])]$. By using our truth assignment h, we can verify that this statement satisfies our characteristic of $J^n_q(\phi)$ such that $J^3_{\frac{1}{2}}(\phi)$ is 1 when ϕ is $\frac{1}{2}$, and 0 otherwise.

2.1 Definition of $\mathcal{D}^n$

Definition 9. *A well-formed formula ϕ is* <u>*n-valued determinate*</u> *if and only if for every n-valued truth assignment h, either $h(\phi)=0$ or $h(\phi)=1$.*

Definition 10. *The set* <u>$\mathcal{D}^n$</u> *is the set of all well-formed formulas built from our $J^n_q(\phi)$s in n-valued sentential logic, using the connectives $\neg$ and $\rightarrow$. We will call such well-formed formulas Boolean combinations of $J^n_q(\phi)$s.*

We can see from our definition of n-valued truth assignment that our extension of a truth assignment h to all well-formed formulas will ensure that Boolean combinations of determinate well-formed formulas will also be determinate. Therefore, $\mathcal{D}^n$ is a set of determinate well-formed formulas, since each $J^n_q(\phi)$ is.

3 Axioms

A_1^n: $(\phi\to\phi)$
A_2^n: $(\phi\to(\psi\to\phi))$
A_3^n: $(\neg\phi\to\neg\psi)\to(\psi\to\phi)$
A_4^n: $(\phi\to(\psi\to\theta))\to((\phi\to\psi)\to(\phi\to\theta))$ for $\phi \in \mathcal{D}^n$
A_5^n: $(J_1^n(\phi)\to\phi)$
A_6^n: $(\underbrace{\phi \to (\phi \to \ldots (\phi}_{(n-1)}\to J_1^n(\phi))\ldots))$
A_7^n: $(\neg J_0^n(\phi)\to(\neg J^n_{\frac{1}{n-1}}(\phi)\to\ldots(\neg J^n_{\frac{n-2}{n-1}}(\phi)\to J_1^n(\phi))\ldots))$
A_8^n: $J_q^n(\phi)\to J_{1-q}^n(\neg\phi)$, for all $q \in \mathbf{T}_n$
A_9^n: $J_q^n(\phi)\to[J_r^n(\psi)\to J^n_{min\{1,1-q+r\}}(\phi \to \psi)]$ for all $q, r \in \mathbf{T}_n$
A_{10}^n: $J_q^n(\phi)\to\neg J_r^n(\phi)$ for all $q \neq$r$\in \mathbf{T}_n$
A_{11}^n: $\neg\phi\to(\phi\to\psi)$
A_{12}^n: $(\phi\to\neg\phi)\to\neg\phi$ for $\phi \in \mathcal{D}^n$

Theorem 3. *All axioms A_1^n through A_{12}^n are n-valued tautologies.*

Further discussion of this theorem can be found in Appendix A.

The axioms here are drawn from the sources used in this paper, specifically the article [**?**] and book [**?**] by Rosser and Turquette, and the article [**?**] by Goldberg, Leblanc, and Weaver. Of note, A_4^n is taken from Mendelson [**?**]. This axiom was extended through a concept of stuttering in Golberg, Leblanc, and Weaver, to 3-valued logic, but it is used in this paper only for determinate sentences. Axioms A_9^n and A_1^n0 were derived from Rosser and Turquette, but are not direct restatements. Finally, A_6^n borrows the concept of stuttering from Goldberg, Leblanc, and Weaver, but applies it to the $J_q^n(\phi)$ concept of Rosser and Turquette. An argument for this axiom can be found in Appendix A.

4 Lemmas

From these axioms, we can derive the following facts:

Lemma 2. *(Finiteness): Let $\Sigma \vdash^n \psi$. Then, there exists a finite subset Σ' of Σ such that $\Sigma' \vdash^n \psi$.*

Proof: Let $\Sigma \vdash^n \psi$ and let ϕ_i, …, ϕ_n be a deduction of ψ from Σ such that $\phi_n = \psi$. Then, for each ϕ_i, $\phi_i \in \Sigma$ or $\phi_i \in \Lambda$ or ϕ_i is the conclusion on an implication, some ϕ_j, ϕ_k, j, k<i. Let $\Sigma' = \{\phi_i : \phi_i \in \Sigma\}$ Then $\Sigma' \vdash^n \psi$ and $|\Sigma'| < \omega$. $\square$

Lemma 3. *Weak Deduction Theorem: If $\Sigma \cup \{\phi\} \vdash^n \psi$, and $\phi \in \mathcal{D}^n$, then $\Sigma \vdash^n (\phi\to\psi)$.*

Proof: Let $\psi_0, \psi_1, \ldots, \psi_m$ be a proof of ψ from $\Sigma \cup \{\phi\}$ with $\psi_m{=}\psi$. We prove by induction on i that $\Sigma{\vdash}^n(\phi{\to}\psi_i)$ for i≤m.
Base case: $\psi_0 \in \Lambda_n$, where Λ_n is the set of logical axioms for the n-valued sentential logic, or $\psi_0{=}\phi$, or $\psi_0 \in \Sigma$. If $\psi_0 \in \Lambda_n$, then $\models^n\psi_0$ and $\Sigma{\vdash}^n\psi_0$. $\Sigma{\vdash}^n\psi_0{\to}(\phi{\to}\psi_0)$ by A_2^n, so $\Sigma{\vdash}^n\phi{\to}\psi_0$ by modus ponens. If $\psi_0{=}\phi$ then since $\Sigma{\vdash}^n\phi{\to}\phi$ by A_1^n, then $\Sigma{\vdash}^n\phi{\to}\psi_0$. If $\psi_0 \in \Sigma$, then $\Sigma{\vdash}^n\psi_0$, and again $\Sigma{\vdash}^n\psi_0{\to}(\phi{\to}\psi_0)$ by A_2^n, so $\Sigma{\vdash}^n\phi{\to}\psi_0$ by modus ponens.

Inductive Hypothesis: Assume for all i<m, $\Sigma{\vdash}^n(\phi{\to}\psi_i)$. We show that $\Sigma{\vdash}^n(\phi{\to}\psi_m)$

We consider 2 cases. Case 1: $\psi \in \Lambda_n$ or $\psi{=}\phi$ or $\psi \in \Sigma$. The proof for these is the same as in the base case. Case 2: ψ_m is the conclusion of an implication from ψ_j and $\psi_k{=}(\psi_j{\to}\psi_m)$ where j,k<m. By the inductive hypothesis, $\Sigma{\vdash}^n\phi{\to}\psi_j$ and $\Sigma{\vdash}^n\psi_k{=}\phi{\to}(\psi_j{\to}\psi_m)$.

By axiom schema A_4^n, $\models^n(\phi{\to}(\psi{\to}\theta)){\to}((\phi{\to}\psi){\to}(\phi{\to}\theta))$ for $\phi \in \mathcal{D}^n$, we have $\Sigma{\vdash}^n(\phi{\to}(\psi_j{\to}\psi_i)){\to}((\phi{\to}\psi_j){\to}(\phi{\to}\psi_i))$. Hence by modus ponens, we get $\Sigma{\vdash}^n((\phi{\to}\psi_j){\to}(\phi{\to}\psi_i))$. And, by modus ponens again, we get $\Sigma{\vdash}^n(\phi{\to}\psi_i)$ as desired. □

Lemma 4. *If a set of well-formed formulas, Σ is inconsistent in n-valued logic, then $\Sigma{\vdash}^n\phi$ for all ϕ in n-valued sentential logic.*

Proof: By definition of inconsistent, $\Sigma{\vdash}^n\phi$ and $\Sigma{\vdash}^n\neg\phi$. From A_{11}^n, we have $\models^n\neg\phi{\to}(\phi{\to}\psi)$. Then, since $\Sigma{\vdash}^n\neg\phi$, and $\Sigma{\vdash}^n\neg\phi{\to}(\phi{\to}\psi)$, we have that $\Sigma{\vdash}^n(\phi{\to}\psi)$ by modus ponens. Since we also know that $\Sigma{\vdash}^n\phi$, we can say $\Sigma{\vdash}^n\psi$ for all ψ, also by modus ponens.

Lemma 5. *If Σ is n-valued consistent, then for any ϕ there exists a $q \in \boldsymbol{T}_n$ such that $\Sigma \cup \{J_q^n(\phi)\}$ is consistent.*

Proof: By way of contradiction, assume $\Sigma \cup \{J_q^n(\phi)\}$ is not consistent for all q. Then by Lemma 4, we can say that $\Sigma \cup \{J_q^n(\phi)\}{\vdash}^n\neg(\delta{\to}\delta)$, for all q, and by the Weak Deduction Theorem, $\Sigma{\vdash}^n\{J_q^n(\phi)\}{\to}\neg(\delta{\to}\delta)$. Therefore, $\Sigma\ {\vdash}^n(\delta{\to}\delta){\to}\neg J_q^n(\phi)$ by A_3^n. Since $\Sigma{\vdash}^n(\delta{\to}\delta)$ by A_1^n, then we can conclude $\Sigma{\vdash}^n\neg J_q^n(\phi)$ by modus ponens. But, A_7^n says
$\models^n(\neg J_0^n(\phi){\to}(\neg J_{\frac{1}{n-1}}^n(\phi){\to}\ldots(\neg J_{\frac{n-2}{n-1}}^n(\phi){\to}J_1^n(\phi))\ldots))$. But, this means that since $\Sigma{\vdash}^n\neg J_q^n(\phi)$, then by modus ponens n-1 times, we get $\Sigma{\vdash}^n J_1^n(\phi)$, but is a contradiction, since we assumed $\Sigma \cup \{J_q^n(\phi)\}$ is inconsistent for all q. Therefore, for any ϕ, there exists a q$\in \mathbf{T}_n$ such that $\Sigma{\cup}J_q^n(\phi)$ is consistent.

Lemma 6. *Let Σ be a set of well-formed formulas and let $\phi \in \mathcal{D}^n$. Then $\Sigma{\cup}\{\phi\}$ is inconsistent iff $\Sigma{\vdash}^n\neg\phi$.*

Proof: First we consider if $\Sigma \vdash^n \neg\phi$. Then $\Sigma \cup \{\phi\} \vdash^n \neg\phi$. But $\Sigma \cup \{\phi\} \vdash^n \neg\phi$ since $\phi \in \Sigma \cup \{\phi\}$. Therefore, $\Sigma \cup \{\phi\}$ is inconsistent. Then, we consider if $\Sigma \cup \{\phi\}$ is inconsistent. Then, by Lemma 4, $\Sigma \cup \{\phi\} \vdash^n \neg\phi$. By the Weak Deduction Theorem, $\Sigma \vdash^n (\phi \rightarrow \neg\phi)$. Then, by $A_1^n 2$ we have $\Sigma \vdash^n (\phi \rightarrow \neg\phi) \rightarrow \neg\phi$. Then, by modus ponens, $\Sigma \vdash^n \neg\phi$. $\square$

5 Strong Completeness

From this point, we will use the axiom schemas, lemmas, and concepts outlined above to prove the Strong Completeness Theorem.

Theorem 4. *Let Σ be a set of well-formed formulas and ϕ be a well-formed formula. Then, $\Sigma \vdash^n \phi$ if and only if $\Sigma \models^n \phi$.*

Soundness, provided above in the definitions and terminology section, gives us one direction of Completeness. For clarity, we restate:

The Soundness Theorem: Let Σ be a set of well-formed formulas and ϕ be a well-formed formula. If $\Sigma \vdash^n \phi$ then $\Sigma \models^n \phi$.

Now, we proceed through a series of steps to prove the remaining direction of Completeness.

5.1 If Σ is consistent in n-valued logic, then Σ can be extended to a maximally consistent Σ_∞ in n-valued logic.

Let Σ be a consistent set of well-formed formulas in n-valued sentential logic, then for every ϕ, either $\Sigma \nvdash^n \phi$ or $\Sigma \nvdash^n \neg\phi$ (or both).

Claim: If Σ is consistent, then Σ can be extended to a maximally consistent Σ_∞.

The well-formed formulas of n-valued sentential logic are countably infinite, and therefore enumerable. Therefore, let the set of all well-formed formulas be $\{\phi_0, \phi_1, \phi_2, \phi_3, \ldots\}$.

Let $\Sigma_0 = \Sigma$ and let

$$\Sigma_{i+1} = \begin{cases} \Sigma_i \cup \{\phi_i\} & \text{if } \Sigma_i \cup \{\phi_i\} \text{ is consistent;} \\ \Sigma_i & \text{Otherwise.} \end{cases}$$

Let $\Sigma_\infty = \bigcup_{i \in \omega} \Sigma_i$.

Claim: Σ_∞ is consistent.

Proof: By way of contradiction, assume Σ_∞ is inconsistent. By the finiteness lemma, there exists a $\Sigma' \subseteq \Sigma_\infty$ such that Σ' is inconsistent as well. But, Σ' must be a subset of some Σ_i, and each Σ_i is consistent by construction, so this is a contradiction. Therefore, Σ_∞ is consistent. $\square$

Claim: Σ_∞ is maximally consistent.

By maximally consistent, we mean that Σ is consistent, and if $\Sigma \cup \{\phi\}$ is consistent, then $\phi \in \Sigma$. By way of contradction, suppose Σ_∞ is not maximally consistent. Then, since we are given that Σ_∞ is consistent, this means that there exists some well-formed formula ϕ such that $\Sigma_\infty \cup \{\phi\}$ is consistent but $\phi \notin \Sigma_\infty$. Since we enumerated our well-formed formulas, we know that ϕ=ϕ_i for some i. Since $\Sigma_\infty \cup \{\phi\}$ is consistent, $\Sigma_i \cup \{\phi\}$ is consistent for every i since $\Sigma_i \cup \{\phi\} \subseteq \Sigma_\infty \cup \{\phi\}$. Then, $\phi \in \Sigma_{i+1}$ by construction, which means that $\phi \in \Sigma_\infty$ which is a contradction. Therefore, Σ_∞ is maximally consistent. $\square$

5.2 If $\Sigma \vdash^n \phi$ then $\Sigma \vdash^n \mathbf{J}_1^n(\phi)$

Proof:
By A_6^n, $\vdash^n(\underbrace{\phi \rightarrow (\phi \rightarrow \ldots (\phi}_{(n-1)} \rightarrow \mathrm{J}_1^n(\phi))\ldots))$, so $\Sigma \vdash^n(\underbrace{\phi \rightarrow (\phi \rightarrow \ldots (\phi}_{(n-1)} \rightarrow \mathrm{J}_1^n(\phi))\ldots))$.
It is given that $\Sigma \vdash^n \phi$. By applying modus ponens n-1 times, we get $\Sigma \vdash^n \mathrm{J}_1^n(\phi)$. $\square$

5.3 If $\phi \in \Sigma_\infty$ then $\mathbf{J}_1^n(\phi) \in \Sigma_\infty$

Proof:

We are given that $\phi \in \Sigma_\infty$, so it follows by the previous section (5.2), that $\Sigma_\infty \vdash^n \mathrm{J}_1^n(\phi)$, and therefore $\Sigma_\infty \cup \{\mathrm{J}_1^n(\phi)\}$ is consistent. Then we can conclude that $\mathrm{J}_1^n(\phi) \in \Sigma_\infty$. $\square$

5.4 Define h which satisfies Σ_∞.

We now proceed by defining a truth assignment h in the following way.

Let S_i be a sentence letter in n-valued sentential logic. Then, define h(S_i)=q if and only if $\mathrm{J}_q^n(S_i) \in \Sigma_\infty$. Then, this definitioin along with our recursive clauses regarding all truth assignments in n-valued logic, define the truth values of all well-formed formulas. We want to show that h(ϕ)=q if and only if $\mathrm{J}_q^n(\phi) \in \Sigma_\infty$.

By our definition of h, $\Sigma_\infty \vdash^n \mathrm{J}_q^n(S_i)$ if and only if h(S_i)=q. We know that since S_i is a sentence letter, the length of S_i, denoted $lh(S_i) = 1$. Therefore, we use induction on the length of ϕ in $\mathrm{J}_q^n(\phi)$. Assume that for any ϕ such that $lh(\phi) < m$, the truth assignment, h(ϕ)=q if and only if $\mathrm{J}_q^n(\phi) \in \Sigma$. Then, there are two cases for $lh(\phi) = m$.

Case 1: ϕ=$\neg\chi$ for some well-formed formula χ. First, we consider if $\mathrm{J}_q^n(\phi) \in \Sigma_\infty$, where ϕ=$\neg\chi$, then h(ϕ)=q. We know that $\Sigma_\infty \vdash^n \mathrm{J}_q^n(\phi)$, so $\Sigma_\infty \vdash^n \mathrm{J}_q^n(\neg\chi)$. By axiom A_8^n, we know that $\models^n \mathrm{J}_q^n(\phi) \rightarrow \mathrm{J}_{1-q}^n(\neg\phi)$, for all $q \in \mathbf{T}_n$. Therefore, we have $\Sigma_\infty \vdash^n \mathrm{J}_{1-q}^n(\chi)$. Since $lh(\chi) < m$, by our inductive hypothesis, we have that h(χ)=1-q, and by our recursive clauses regarding all truth assignments, we have that h($\neg\chi$)=1-(1-q)=q. Therefore, h(ϕ)=q.

Now, we consider if h(ϕ)=q, where ϕ=$\neg\chi$, then $\mathrm{J}_q^n(\phi) \in \Sigma_\infty$. Since h($\neg\chi$)=q, we know that h($\chi$)=1-q, and since $lh(\chi) < m$, by our inductive hypothesis, we have that $\Sigma_\infty \vdash^n J^n_{1-q}(\chi)$. Then, by axiom A^n_8 and by A^n_3, we have that $\Sigma_\infty \vdash^n J^n_q(\neg\chi)$, and therefore, $\Sigma_\infty \vdash^n J^n_q(\phi)$.

Case 2: Let $\phi=\chi\rightarrow\mu$ for some well-formed formulas χ and μ. Since $lh(\phi)$=m, we know that $lh(\chi)$ and $lh(\mu)$ are less than m. As above, we consider first if $J^n_q(\phi)\in\Sigma_\infty$, where $\phi=\chi\rightarrow\mu$, then h(ϕ)=q. We know that $\Sigma_\infty\vdash^n J^n_q(\phi)$, so $\Sigma_\infty\vdash^n J^n_q(\chi\rightarrow\mu)$. By Lemma 5, we can see that there exists an r such that $J^n_r(\chi)\in \Sigma_\infty$ and an s such that $J^n_s(\mu)\in \Sigma_\infty$. Then, by section 5.3, if $J^n_r(\chi)\in \Sigma_\infty$ and $J^n_s(\mu)\in \Sigma_\infty$, then $\Sigma_\infty\vdash^n J^n_r(\chi)$ and $\Sigma_\infty\vdash^n J^n_s(\mu)$. Axiom A^n_9 states
$\models^n J^n_q(\phi)\rightarrow[J^n_r(\psi)\rightarrow J^n_{min\{1,1-q+r\}}(\phi \rightarrow \psi)]$, which means that
$\Sigma_\infty\vdash^n J^n_r(\chi)\rightarrow[J^n_s(\mu)\rightarrow J^n_{min\{1,1-r+s\}}(\chi \rightarrow \mu)]$. Since we know that $\Sigma_\infty\vdash^n J^n_r(\chi)$, then by modus ponens we have that $\Sigma_\infty\vdash^n J^n_s(\mu)\rightarrow J^n_{min\{1,1-r+s\}}(\chi \rightarrow \mu)$. By modus ponens again, since $\Sigma_\infty\vdash^n J^n_s(\mu)$, we have $\Sigma_\infty\vdash^n J^n_{min\{1,1-r+s\}}(\chi \rightarrow \mu)$. It also follows by our inductive hypothesis that h(χ)=r and h(μ)=s, and, by our recursive clauses regarding truth assignments, that h($\chi\rightarrow\mu$)=h(ϕ)=min{1, 1-r+s}. But, we said that
$\Sigma_\infty\vdash^n J^n_{min\{1,1-r+s\}}(\chi \rightarrow \mu)$ or $\Sigma_\infty\vdash^n J^n_{min\{1,1-r+s\}}(\phi)$. Thus, if $J^n_q(\phi)\in\Sigma_\infty$, where $\phi=\chi\rightarrow\mu$, then h(ϕ)=q.

Now, we consider if h(ϕ)=q, where $\phi=\chi\rightarrow\mu$, then we want to show $J^n_q(\phi)\in\Sigma_\infty$. Since h($\phi$)= h($\chi\rightarrow\mu$)=q, we know that h($\chi\rightarrow\mu$)=min{1, 1-h($\chi$)+h($\mu$)}=q, and since $lh(\chi) < m$ and $lh(\mu) < m$, by our inductive hypothesis, we have that h(χ)=r $\leftrightarrow$ $J^n_r(\chi)\in \Sigma_\infty$ and h(μ)=s $\leftrightarrow$ $J^n_s(\mu)\in \Sigma_\infty$. Then, $\Sigma_\infty\vdash^n J^n_r(\chi)$ and $\Sigma_\infty\vdash^n J^n_s(\mu)$, so using Axiom A^n_9, we get that
$\Sigma_\infty\vdash^n J^n_r(\chi)\rightarrow[J^n_S(\mu)\rightarrow J^n_{min\{1,1-r+s\}}(\chi \rightarrow \mu)]$. Then, by modus ponens twice, we have that $\Sigma_\infty\vdash^n J^n_{min\{1,1-r+s\}}(\chi \rightarrow \mu)]$. So, $\Sigma_\infty\vdash^n J^n_{min\{1,1-r+s\}}(\phi)]$. So, $\Sigma_\infty\vdash^n J^n_q(\phi)]$, and therefore, we conclude that $J^n_q(\phi)\in \Sigma_\infty$ by construction of Σ_∞.

Therefore, we have shown that $J^n_q(\phi)\in\Sigma_\infty$ if and only if h(ϕ)=q. □

5.5 If Σ is consistent then Σ is satisfiable.

Proof: We have shown that for Σ consistent, For all $\phi \in \Sigma$ we have that $J^n_1(\phi)\in \Sigma_\infty$. This means that our new truth assignment h assigns the truth value of 1 to $J^n_1(\phi)$ and also to ϕ. Then, we can say that h($J^n_1(\phi)$)=1 and h(ϕ)=1 for all $\phi \in \Sigma$. Thus, by the definition of satisfiable, Σ is satisfiable.

5.6 Strong Completeness.

We assume $\Sigma \models^n \phi$. We will show that $\Sigma \vdash^n \phi$. If $\Sigma \models^n \phi$ then $\Sigma \models^n J^n_1(\phi)$. Therefore $\Sigma\cup\{\neg J^n_1(\phi)\}$ is not satisfiable. Then, by using the result from section 5.5, $\Sigma\cup\{\neg J^n_1(\phi)\}$ is not consistent. Therefore, by Lemma 4, $\Sigma\cup\{\neg J^n_1(\phi)\}\vdash^n \neg(\psi\rightarrow\psi)$. And, by the Weak Deduction Theorem, we have that $\Sigma\vdash^n \neg J^n_1(\phi)\rightarrow \neg(\psi\rightarrow\psi)$. Then, by applying A^n_3, $\Sigma\vdash^n(\psi\rightarrow\psi)\rightarrow J^n_1(\phi)$. And then we have that $\Sigma\vdash^n J^n_1(\phi)$ by modus ponens. And finally, by applying A^n_5 and modus ponens, we get $\Sigma\vdash^n\phi$. □

Therefore, we have that if Σ is a set of well-formed formulas and ϕ is a well-formed formula. Then, $\Sigma \vdash^n \phi$ if and only if $\Sigma \models^n \phi$.

6 Conclusion

We have shown that using our set of axioms schemas, we can prove Strong Completeness in n-valued sentential logic. While this set of axioms is not minimal, for any set of axioms that includes all of these, Strong Completeness will hold. And, further, for any set of axioms from which all of these are deducible, Strong Completeness will hold, without further work.

7 Appendix A: Discussion of Theorem 3

Each of the axioms provided in section 3 of this paper is an n-valued tautology. Arguments are provided here for select axioms to verify this point.

We show $\models^n(\phi\rightarrow(\psi\rightarrow\phi))$ [A_2^n]

Argument: We recall that h($\phi \rightarrow \psi$)=1 if and only if h(ϕ)$\leq$h(ψ). Therefore, we want to show that h(ϕ)$\leq$h($\psi\rightarrow\phi$). h($\psi \rightarrow \phi$)=min$\{1$, 1-h(ψ)+h(ϕ)$\}$ by the definition of our truth assignment h. So, we need to show that h(ϕ)$\leq$min$\{1$, 1-h(ψ)+h(ϕ)$\}$. h(ϕ) must be either less than, equal to, or greater than h(ψ), so we consider those three cases.

Case 1: h(ϕ)<h(ψ). Then, 1-h(ψ)+h(ϕ)<1 and so min$\{1$, 1-h(ψ)+h(ϕ)$\}$=1-h(ψ)+h(ϕ). So, our inequality is h(ϕ)$\leq$1-h(ψ)+h(ϕ), which is necessarily true. Therefore, our claim holds for this case.

Case 2: h(ϕ)=h(ψ). Then, 1-h(ψ)+h(ϕ)=1 and so min$\{1$, 1-h(ψ)+h(ϕ)$\}$=1. So, our inequality is h(ϕ)$\leq$1, which is true by the definition of our truth assignment h. Therefore, our claim holds for this case.

Case 3: h(ϕ)>h(ψ). Then, 1-h(ψ)+h(ϕ)>1 and so min$\{1$, 1-h(ψ)+h(ϕ)$\}$=1. So, again our inequality is h(ϕ)$\leq$1, which is true by the definition of our truth assignment h. Therefore, our claim holds for this case.

So, A_2^n holds for n-valued sentential logic.$\Box$

We show $\models^n(\neg\phi\rightarrow\neg\psi)\rightarrow(\psi\rightarrow\phi)$ [A_3^n]

Argument: Let h(ϕ)=k and let h(ψ)=l. Then h($\neg\phi$)=1-k and h($\neg\psi$)=1-l. Then h($\neg\phi \rightarrow\neg\psi$)= min$\{1$, 1-(1-k)+(1-l)$\}$=min$\{1$, 1-l+k$\}$ and h($\psi \rightarrow \phi$)=min$\{1$, 1-l+k$\}$. Since these are the same, regardless of the values of k and l, h($\neg\phi\rightarrow\neg\psi$)$\rightarrow$($\psi \rightarrow \phi$)=1.

So, claim A_3^n holds for n-valued sentential logic. $\Box$

We show $\models^n(\phi\rightarrow(\psi\rightarrow\theta))\rightarrow((\phi\rightarrow\psi)\rightarrow(\phi\rightarrow\theta))$ for $\phi \in \mathcal{D}^n$ [A_4^n]

Argument: Using our definition of the truth assignment h, we see that h((ϕ→(ψ→θ))→((ϕ→ψ)→(ϕ→θ))) =min{1, 1-h(ϕ→(ψ→θ))+h((ϕ→ψ)→(ϕ→θ))}. Because $\phi \in \mathcal{D}^n$, we know that h(ϕ) can only equal 0 or 1. Therefore, we look at those two cases. Case 1: if h(ϕ)=1, then h(ϕ→(ψ→θ))= h(ψ→θ), h(ϕ→ψ)=h(ψ), and h(ϕ→θ)=h(θ). So, h((ϕ→(ψ→θ))→((ϕ→ψ)→(ϕ→θ))) = min{1, 1-h(ψ→θ)+ h((ϕ→ψ)→(ϕ→θ))} = min{1, 1-min{1,1-h(ψ)+h(θ)}+min{1, 1-h(ϕ→ψ)+h(ϕ→θ)}} = min{1, 1-min{1,1-h(ψ)+h(θ)}+min{1, 1-h(ψ)+h(θ)}}=1. Case 2: if h(ϕ)=0, then h(ϕ→(ψ→θ))= 1 h(ϕ→ψ)=1, and h(ϕ→θ)=1. So, h((ϕ→(ψ→θ))→((ϕ→ψ)→(ϕ→θ))) =min{1, 1-1+min{1, 1-1+1}}=min{1, 1-1+1}=1.}□

We show $\models^n(\underbrace{\phi \to (\phi \to \ldots (\phi}_{(n-1)}\to J_1^n(\phi))\ldots))$ [A_6^n]

Argument: We have stated that h(ϕ→ψ)=min{1, 1-h(ϕ)+h(ψ)}. Then we can say that h$(\underbrace{\phi \to (\phi \to \ldots (\phi}_{(n-1)}\to J_1^n(\phi))\ldots))$ =

min{1, 1 - h(ϕ) + h$(\underbrace{\phi \to (\phi \to \ldots (\phi}_{(n-2)}\to J_1^n(\phi))\ldots))$}, and thus

h$(\underbrace{\phi \to (\phi \to \ldots (\phi}_{(n-1)}\to J_1^n(\phi))\ldots))$= min{1, (n-1)(1-h($\phi$))+h($J_1^n(\phi)$)}. We also know that 0≤h(ϕ)≤1 for all ϕ, and we have stated that for a specific value of n, the set of truth values $\mathbf{T}_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, and so h($\phi$)∈ $\mathbf{T}_n$. Therefore, h(ϕ) = $\frac{b}{n-1}$ for some 0≤b≤n-1. If b=n-1, then h(ϕ) = 1 and then h($J_1^n(\phi)$) = 1, so h$(\underbrace{\phi \to (\phi \to \ldots (\phi}_{(n-1)}\to J_1^n(\phi))\ldots))$ = 1. If b<n-1, then h($J_1^n(\phi)$) = 0, since h($J_1^n(\phi)$) = 0 if h(ϕ)≠1. Then, min{1, (n-1)(1-h(ϕ))+h($J_1^n(\phi)$)} = min{1, (n-1)(1-$\frac{b}{n-1}$)+h($J_1^n(\phi)$)} = min{1, ((n-1)-b)+h($J_1^n(\phi)$)}. But, since b<n-1, n-1-b≥1, so min{1, (n-1)(1-h(ϕ))+h($J_1^n(\phi)$)} = 1. Therefore,
h$(\underbrace{\phi \to (\phi \to \ldots (\phi}_{(n-1)}\to J_1^n(\phi))\ldots))$ = 1.

We show $\models^n(\phi\to\neg\phi)\to\neg\phi$ for $\phi \in \mathcal{D}^n$ [A_1^n2]

Argument: We have stated that h($\neg\phi$) = 1-h(ϕ) for all ϕ, and h(ϕ→ψ) = min{1, 1-h(ϕ)+h(ψ)}. Therefore, we can see that if $\phi \in \mathcal{D}^n$, then there are two cases. Case 1: h(ϕ) = 1, then h((ϕ→¬ϕ)→¬ϕ) = 1. Case 2: h(ϕ) = 0, then h((ϕ→¬ϕ)→¬ϕ) = 1. Thus, $\models^2(\phi\to\neg\phi)\to\neg\phi$ is a two-valued tautology.

8 Works Cited

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